Integrating Factors: Step-by-Step

Given any first order linear differential equation, use the following process.

1. Put the equation into standard form: $y' + p(t)y = q(t)$
2. Determine the integrating factor, $\mu = e^{\int p(t)dt}$
3. Multiply both sides of the equation by μ
4. Replace the left side of the equation with $\dfrac{d}{dt}(\mu y)$
5. Multiply both sides by dt and integrate both sides
6. Solve for y
7. If given an initial condition, plug it in and solve for C

Example 1

Solve the differential equation by the method of integrating factors.

$$y' + \tfrac{1}{t}y = t$$

Step 1: Put the equation into standard form

This equation is already in standard form.

Step 2: Determine the integrating factor

$$\begin{aligned}\mu &= e^{\int p(t)dt}\\ &= e^{\int \frac{1}{t}dt}\\ &= e^{\ln|t|}\\ &= t\end{aligned}$$

Step 3: Multiply both sides of the equation by μ

$$ty' + y = t^2$$

Step 4: Replace the left side of the equation with $\dfrac{d}{dt}(\mu y)$

$$\frac{d}{dt}(ty) = t^2$$

Step 5: Multiply both sides by dt and integrate both sides

$$\begin{aligned}d(ty) &= t^2dt\\ \int d(ty) &= \int t^2dt\\ ty &= \frac{1}{3}t^3 + C\end{aligned}$$

Step 6: Solve for y

$$y = \tfrac{1}{3}t^2 + Ct^{-1}$$

As there was no initial condition given, we are finished, and this is our general solution.

Example 2

Solve by method of integrating factors:

$$\frac{dr}{d\theta} + \frac{1}{\theta^2+1}r = \frac{1}{e^{\tan^{-1}\theta}}$$
$$r(0) = \pi$$

First of all, notice that our variables are different from what we are used to. Remember, however, that the variables in the problem do not change the method of solution. You just have to identify the independent and dependent variables, which is given by the differential notation $\frac{d*}{d*}$, as follows:

$$\frac{d(\text{dependent})}{d(\text{independent})}$$

So, in this problem, r is the dependent variable and θ is the independent variable.

We proceed in similar fashion to Example 1.

Step 1: Put the equation into standard form

This equation is already in standard form.

Step 2: Determine the integrating factor

$$\begin{aligned}\mu &= e^{\int p(t)dt} \\ &= e^{\int \frac{1}{\theta^2+1}d\theta} \\ &= e^{\tan^{-1}\theta}\end{aligned}$$

Step 3: Multiply both sides of the equation by μ

$$e^{\tan^{-1}\theta}\frac{dr}{d\theta} + \frac{e^{\tan^{-1}\theta}}{\theta^2+1}r = 1$$

Step 4: Replace the left side of the equation with $\frac{d}{dt}(\mu y)$

$$\frac{d}{d\theta}(e^{\tan^{-1}\theta}r) = 1$$

Step 5: Multiply both sides by dt and integrate both sides

$$\begin{aligned}d(e^{\tan^{-1}\theta}r) &= d\theta \\ \int d(e^{\tan^{-1}\theta}r) &= \int d\theta \\ e^{\tan^{-1}\theta}r &= \theta + C\end{aligned}$$

Step 6: Solve for r

$$r = (\theta + C)e^{-\tan^{-1}\theta}$$

Step 7: If given an initial condition, plug it in and solve for C

Our initial condition is $r(0) = \pi$.

$$\begin{aligned}r(0) = \pi &= (0 + C)e^{-\tan^{-1}(0)} \\ \pi &= Ce^0 = C\end{aligned}$$

Plugging this C into the equation from Step 6, we get our solution

$$r = (\theta + \pi)e^{-\tan^{-1}\theta}$$

Example 3

Solve the differential equation:

$$t^2 y' - 2t^3 y = 4t^3$$
$$y(1) = 0$$

Step 1: Put the equation into standard form

This equation is *not* in standard form. In order to get it in standard form, we merely need to divide by the coefficient of y'. Dividing both sides of the equation by t^2 yields

$$y' - 2ty = 4t$$

Step 2: Determine the integrating factor

$$\begin{aligned} \mu &= e^{\int p(t)dt} \\ &= e^{\int -2t dt} \\ &= e^{-t^2} \end{aligned}$$

Step 3: Multiply both sides of the equation by μ

$$e^{-t^2} y' - 2te^{-t^2} y = 4te^{-t^2}$$

Step 4: Replace the left side of the equation with $\frac{d}{dt}(\mu y)$

$$\frac{d}{dt}(e^{-t^2} y) = 4te^{-t^2}$$

Step 5: Multiply both sides by dt and integrate both sides

$$\begin{aligned} d(e^{-t^2} y) &= 4te^{-t^2} dt \\ e^{-t^2} y &= \int 4te^{-t^2} dt \\ &= -2e^{-t^2} + C \end{aligned}$$

Step 6: Solve for y

$$y = -2 + Ce^{t^2}$$

Step 7: If given an initial condition, plug it in and solve for C

Using the condition, $y(1) = 0$, we have

$$\begin{aligned} 0 &= -2 + Ce^1 \\ 2 &= Ce \\ 2e^{-1} &= C \end{aligned}$$

Plugging this back into the solution from Step 6, we get

$$\begin{aligned} y &= -2 + (2e^{-1})e^{t^2} \\ &= -2 + 2e^{t^2-1} \end{aligned}$$

Example 4

Solve the given differential equation.

$$ty' + (t-1)y = t^2$$

Step 1: Put the equation into standard form

Dividing both sides of the equation by t, we have

$$y' + \frac{t-1}{t}y = t$$

Step 2: Determine the integrating factor

$$\begin{aligned}
\mu(t) &= e^{\int p(t)dt} \\
&= e^{\int \frac{t-1}{t}dt} \\
&= e^{\int (1-\frac{1}{t})dt} \\
&= e^{t-\ln t} \\
&= e^t e^{-\ln t} \\
&= \frac{e^t}{t}
\end{aligned}$$

Step 3: Multiply both sides of the equation by μ

$$\frac{e^t}{t}y' + \frac{e^t(t-1)}{t^2} = e^t$$

Step 4: Replace the left side of the equation with $\frac{d}{dt}(\mu y)$

$$\frac{d}{dt}\left(\frac{e^t}{t}y\right) = e^t$$

Step 5: Multiply both sides by dt and integrate both sides

$$\begin{aligned}
d\left(\frac{e^t}{t}y\right) &= e^t dt \\
\int d\left(\frac{e^t}{t}y\right) &= \int e^t dt \\
\frac{e^t}{t}y &= e^t + C
\end{aligned}$$

Step 6: Solve for y

$$\begin{aligned}
y &= t + \frac{Ct}{e^t} \\
y &= t + Cte^{-t}
\end{aligned}$$

Step 7: If given an initial condition, plug it in and solve for C

There is no initial condition, so that is our solution.